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ALMOST-BALANCED STRUCTURAL DYNAMICS

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Analytical models and experimentally derived models are often represented in the modal co-ordinates. The modal models include natural modes and their displacements. The scaling of the modes is arbitrary, and thus the scaling factor can be freely chosen. By the proper choice of the scaling factor one obtains controllability and observability grammians of a structure that are almost equal, and the system model is almost balanced. This new model is given either in the form of a second order differential equation or in state space representation. The properties of the almost-balanced modes and structures are derived, including its H_2, H_∞ and Hankel norms. The norms of the modes and of the system are expressed in terms of the modal parameters, such as natural frequencies, modal damping, and input and output gains. The properties of the almost balanced structure are used to reduce the model. In particular, the H_2, H_∞ and Hankel system norms are used to evaluate and to minimize the reduction error. Next, the method of placement of actuators or sensors in the almost-balanced co-ordinates is presented. The norm of the almost-balanced mode with a set of actuators (or sensors) is the root-mean-square sum of the norms of this mode for each single actuator (or sensor). Using this property one finds a specified number of actuator or sensor locations such that the system performance evaluated at these locations is close to that of the system with a larger set of candidate locations.

1. INTRODUCTION

The properties of a structural model depend on the co-ordinates chosen for its representation. The most common, but not always the most convenient, are the nodal co-ordinates, which consist of the nodal displacements and velocities. Structural analysts, as well as test engineers, prefer to deal with modal models. The physical accessibility of the modal variables and the independence of the modes make the modal approach suitable for measurements and is convenient in analysis. However, the modes are scaled arbitrarily, and thus the modal models are not unique. The system matrices, such as the natural frequency matrix and the modal damping matrix, are invariant under the modal scaling, while the input and output matrices depend on it. By having the freedom to choose the scaling parameter, one can determine it such that the mode controllability and observability measures are almost identical. In this case the obtained system is almost balanced. In this paper the almost-balanced co-ordinates are introduced. The almost-balanced structure has several useful properties that allow it to be reduced in an almost optimal manner, and allocate the sensors or actuators in the way that the system norm is maximized.

Model reduction techniques are important tools in system dynamics and control. The order of the dynamic system model should be small, but should cause small output errors when compared to the full model output. The "compactness" of a model is particularly important in controller design, where often the controller order (and, of course, its complexity) depends on the system order. Many reduction techniques have already been

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developed. Reduction methods such as those of Hyland and Bernstein [1] and of Wilson [2, 3], give optimal results, but they are complex and computationally expensive. Other, non-optimal methods include balanced and modal truncation; see Moore [4] and Skelton [5, 6]. Model reduction of flexible structures is more specific, and it has been studied by Gregory [7], Jonckheere [8], Skelton [6], Gawronski and Juang [9], Gawronski and Williams [10] and Gawronski [11]. In this paper we present the near-optimal truncation approach in almost-balanced co-ordinates, in terms of H_2, H_∞ and Hankel norms.

The problem of placing a small number of actuators or sensors such that their performance would be close to the larger set of actuators or sensors has obvious practical applications. Typically, the actuator/sensor placement problem is computationally complex because it requires a search and evaluation of a large number of placement cases; see, for example, Aidarous *et al*. [12], Basseville *et al*. [13], DeLorenzo [14], Kammer [15], Kim and Junkins [16], Lim [17], Lim *et al*. [18], Lim and Gawronski [19], Lindberg and Longman [20], Longman and Alfriend [21], Maghami and Joshi [22] and Skelton and DeLorenzo [23]. Here we use the properties of the almost-balanced representation to develop a simple method of placement which gives the performance of a small set of actuators (sensors) that is close to the performance of the initial large set.

2. BALANCED SYSTEMS

Consider a linear, stable, observable and controllable system with the state space representation (A, B, C). The controllability and observability grammians are the stationary solutions of the Lyapunov equations

$$AW_c + W_cA^{\mathrm{T}} + BB^{\mathrm{T}} = 0, \qquad A^TW_o + W_oA + C^{\mathrm{T}}C = 0. \tag{1}$$

For stable A the solutions W_c and W_o are positive definite. The grammians characterize the controllability and observability properties of a system. The system triple is balanced if its controllability and observability grammians are equal and diagonal, as defined by Moore [4]: $W_c = W_o = \Gamma^2$, $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$, $\gamma_i \geqslant 0$, $i = 1, \ldots, n$, where the positive variable γ_i is the ith Hankel singular value of the system.

3. DEFINITION OF FLEXIBLE STRUCTURE

A flexible structure is a linear system represented by the following second order matrix differential equation:

$$M\ddot{q} + D\dot{q} + Kq = B_ou, \qquad y = C_{oq}q + C_{ov}\dot{q}. \tag{2}$$

In this equation q is the $n_d \times 1$ displacement vector, u is the $s \times 1$ input vector, y is the output vector, $r \times 1$, M is the mass matrix, $n_d \times n_d$, D is the proportional damping matrix, $n_d \times n_d$ (see Meirovitch [24] for the definition of the proportional damping), K is the stiffness matrix, $n_d \times n_d$, the input matrix B_o is $n_d \times s$, the output displacement matrix C_{oq} is $r \times n_d$, and the output velocity matrix C_{ov} is $r \times n_d$. The number n_d is the number of degrees of freedom of the system (linearly independent co-ordinates describing the finite-dimensional structure), r is the number of outputs, and s is the number of inputs. The mass matrix is positive definite, and the stiffness and damping matrices are positive semi-definite.

4. SECOND ORDER MODELS

Depending on the chosen co-ordinates, the structural second order equation (2) can be presented in different forms. Furthermore, we consider the modal and the almost-balanced co-ordinates, and related modal and almost-balanced second order models.

4.1. MODAL MODEL

Modal co-ordinates are willingly used, since the displacements in these co-ordinates are independent, and the modes, which define the modal transformation, can be easily measured. Let $\Phi(n_d \times p)$ be the modal matrix, which consists of p natural modes of a structure, $p \leqslant n_d$. Using this matrix one obtains the diagonal modal mass, stiffness and damping matrices $M_m = \Phi^{\mathrm{T}} M \Phi$, $K_m = \Phi^{\mathrm{T}} K \Phi$ and $D_m = \Phi^{\mathrm{T}} D \Phi$.

4.1.1. *System representation*

Introduce a new variable q_m, such that $q = \Phi q_m$. Substituting q in equation (2) one obtains the modal model as follows:

$$\ddot{q}_m + 2Z\Omega\dot{q}_m + \Omega^2 q_m = B_m u, \qquad y = C_{mq} q_m + C_{mv}\dot{q}_m, \tag{3}$$

where $Z = M_m^{-1/2} K_m^{-1/2} D_m$ is the matrix of modal damping, and $\Omega = M_m^{-1} K_m$ is a diagonal matrix of natural frequencies $\Omega = \mathrm{diag}\,(\omega_1, \omega_2, \ldots, \omega_p)$; B_m is the modal input matrix, $B_m = M_m^{-1}\Phi^{\mathrm{T}} B_o$, and C_{mq}, C_{mv} are the modal displacement and rate output matrices, respectively; $C_{mq} = C_{oq}\Phi$ and $C_{mv} = C_{ov}\Phi$.

Note that the second order modal model has two output matrices. We introduce an equivalent output matrix, C_m, as a combination of the output matrices C_{mq} and C_{mv}:

$$C_m = C_{mq}\Omega^{-1} + C_{mv}. \tag{4}$$

The two-norm of C_m is called a system gain. It has the property

$$\|C_m\|_2^2 = \|C_{mq}\Omega^{-1}\|_2^2 + \|C_{mv}\|_2^2, \tag{5}$$

where $\|X\|_2$ is the Euclidean norm of X, $\|X\|_2 = \mathrm{tr}\,(XX^{\mathrm{T}})$. With the output matrix defined, the model equation (3) is alternatively presented as a quadruple (Ω, Z, B_m, C_m) and is called a modal second order representation.

4.1.2. *Mode representation*

The modal equation (3) can be written as a set of p independent equations for each model displacement

$$\ddot{q}_{mi} + 2\zeta_i\omega_i\dot{q}_{mi} + \omega_i^2 q_{mi} = b_{mi} u, \quad y_i = c_{mqi} q_{mi} + c_{mvi}\dot{q}_{mi}, \qquad i = 1, \ldots, p, \tag{6}$$

and c_{mi} is defined as the ith mode equivalent output matrix

$$c_{mi} = \frac{c_{mqi}}{\omega_i} + c_{mvi}. \tag{7}$$

In the above equations, y_i is the system output due to the ith mode dynamics, while b_{mi} is the ith row of B_m, and c_{mqi}, c_{mvi} and c_{mi} are the ith columns of C_{mq}, C_{mv} and C_m, respectively. The quadruple $(\omega_i, \zeta_i, b_{mi}, c_{mi})$, corresponding to equation (6), is the representation of the natural mode, while $\|b_{mi}\|_2$ and $\|c_{mi}\|_2$ are the input and output gains of the ith mode. It is easy to see that the system gains are the r.m.s. sum of the modal gains:

$$\|B_m\|_2 = \sqrt{\sum_{i=1}^{p} \|b_{mi}\|_2^2}, \qquad \|C_m\|_2 = \sqrt{\sum_{i=1}^{p} \|c_{mi}\|_2^2}. \tag{8}$$

4.2. ALMOST-BALANCED MODEL

Define a diagonal transformation matrix $R = \text{diag}\,(r_i)$, $i = 1, \ldots, p$, where the ith entry r_i is a root of the ratio of the modal input and output gains,

$$r_i = \sqrt{\|b_{mi}\|_2 / \|c_{mi}\|_2}. \tag{9}$$

The scaled natural modes, $\phi_{abi} = r_i \phi_i$, $i = 1, \ldots, p$, are called the almost-balanced modes. The almost-balanced mode matrix $\Phi_{ab} = [\phi_{ab1} \;\; \phi_{ab2} \;\; \cdots \;\; \phi_{abp}]$ is obtained from the generic modal matrix as

$$\Phi_{ab} = \Phi R. \tag{10}$$

This matrix diagonalizes the mass and the stiffness matrices

$$\Phi_{ab}^{\mathrm{T}} M \Phi_{ab} = R^2 M_m, \qquad \Phi_{ab}^{\mathrm{T}} K \Phi_{ab} = R^2 K_m. \tag{11}$$

4.2.1. *System representation*

The almost-balanced model is the one with the displacement q_{ab} obtained by scaling the modal displacement q_m; i.e., $q_m = R q_{ab}$. By introducing this equation to the modal equation (3) one obtains the almost-balanced second order model

$$\ddot{q}_{ab} + 2Z\Omega \dot{q}_{ab} + \Omega^2 q_{ab} = B_{ab} u, \qquad y = C_{abq} q_{ab} + C_{abv} \dot{q}_{ab}, \tag{12}$$

where

$$B_{ab} = R^{-1} B_m, \qquad C_{abq} = C_{mq} R, \qquad C_{abv} = C_{mv} R. \tag{13}$$

The equivalent output matrix C_{ab} of the almost-balanced model is a combination of the output matrices C_{abq} and C_{abv}:

$$C_{ab} = C_{abq} \Omega^{-1} + C_{abv}. \tag{14}$$

It has the property

$$\|C_{ab}\|_2^2 = \|C_{abq} \Omega^{-1}\|_2^2 + \|C_{abv}\|_2^2. \tag{15}$$

The equation (12) is alternatively presented as a quadruple $(\Omega, Z, B_{ab}, C_{ab})$ and is called the almost-balanced second order representation.

4.2.2. *Mode representation*

Since the displacement of the ith almost-balanced mode q_{abi} is

$$q_{abi} = q_{mi} / r_i, \tag{16a}$$

equation (12) can be written as a set of p independent equations for each almost-balanced displacement:

$$\ddot{q}_{abi} + 2\zeta_i \omega_i \dot{q}_{abi} + \omega_i^2 q_{abi} = b_{abi} u, \quad y_i = c_{abqi} q_{abi} + c_{abvi} \dot{q}_{abi}, \qquad i = 1, \ldots, p, \tag{16b}$$

and c_{abi} is the ith mode equivalent output matrix

$$c_{abi} = \frac{c_{abqi}}{\omega_i} + c_{abvi}. \tag{17}$$

In the above equations, y_i is the system output due to the ith almost-balanced mode dynamics, and b_{abi} is the ith row of B_{ab}, and c_{abqi}, c_{abvi} and c_{abi} are ith columns of C_{abq}, C_{abv} and C_{ab}, respectively. The quadruple $(\omega_i, \zeta_i, b_{abi}, c_{abi})$ corresponding to equation (16) is the representation of the almost-balanced mode.

Define $\|b_{abi}\|_2$ as the input gain of the ith almost-balanced mode, and $\|c_{abi}\|_2$ as the output gain of the same mode. In the second order almost-balanced model, the input and output gains are equal:

$$\|b_{abi}\|_2 = \|c_{abi}\|_2. \tag{18}$$

In order to prove this, the transformation R as in equation (9) is introduced to equations (13) and (14):

$$\|b_{abi}\|_2 = \|b_{mi}\|_2/\|r_i\|_2 = \|b_{mi}\|_2\sqrt{\|c_{mi}\|_2/\|b_{mi}\|_2} = \sqrt{\|b_{mi}\|_2\|c_{mi}\|_2}, \tag{19a}$$

$$\|c_{abi}\|_2 = \|c_{mi}\|_2 r_i = \|c_{mi}\|_2\sqrt{\|b_{mi}\|_2/\|c_{mi}\|_2} = \sqrt{\|b_{mi}\|_2\|c_{mi}\|_2}, \tag{19b}$$

which shows that equation (18) holds.

The relationship between the modal, balanced and almost-balanced representation is illustrated in Figure 1. In this figure almost-balanced co-ordinates are co-linear with the modal co-ordinates, and are almost identical with the balanced co-ordinates.

4.2.3. *Controllability and observability grammians*

The controllability and observability grammians are defined in the state space representation, and in general cannot be obtained explicitly from the second order model. However, we will show that the approximate grammians can be obtained from the almost-balanced second order model. Moreover, in this model the grammians are almost equal and diagonal, thus the model is almost balanced.

The controllability (w_c) and observability (w_o) grammians for the second order almost balanced model are given as

$$w_c = 0{\cdot}25Z^{-1}\Omega^{-1}\,\mathrm{diag}\,(B_{ab}B_{ab}^{\mathrm{T}}), \qquad w_o = 0{\cdot}25Z^{-1}\Omega^{-1}\,\mathrm{diag}\,(C_{ab}^{\mathrm{T}}C_{ab}), \tag{20}$$

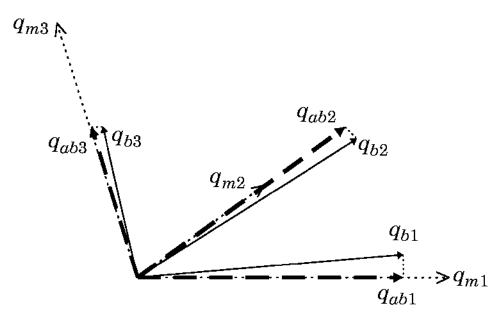

Figure 1. Modal, balanced, and almost-balanced co-ordinates.

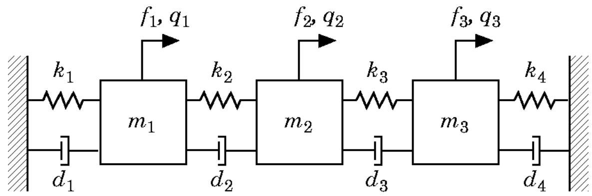

Figure 2. A simple system.

where diag $(B_{ab}B_{ab}^T)$ and diag $(C_{ab}^T C_{ab})$ denote the diagonal parts of $B_{ab}B_{ab}^T$ and $C_{ab}^T C_{ab}$, respectively. Therefore, the ith diagonal entries of w_c and w_o are

$$w_{ci} = \frac{\|b_{abi}\|_2^2}{4\zeta_i\omega_i}, \qquad w_{oi} = \frac{\|c_{abi}\|_2^2}{4\zeta_i\omega_i}, \tag{21}$$

In order to prove this, define the following state vector:

$$x_{ab} = \begin{bmatrix} \Omega q_{ab} \\ \dot{q}_{ab} \end{bmatrix} \tag{22}$$

associated with the following state space representation

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -2Z\Omega \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_m \end{bmatrix}, \qquad C = [C_{mq}\Omega^{-1} \quad C_{mv}]. \tag{23}$$

By inspection, for this representation the grammians are diagonally dominant, in the form

$$W_c \simeq \begin{bmatrix} w_c & 0 \\ 0 & w_c \end{bmatrix}, \qquad W_o \simeq \begin{bmatrix} w_o & 0 \\ 0 & w_o \end{bmatrix}, \tag{24}$$

where w_c and w_o are the diagonally dominant matrices, $w_c \simeq$ diag (w_{ci}) and $w_o \simeq$ diag (w_{oi}). Introducing equations (23) and (24) to the Lyapunov equations (1), one obtains equation (20).

We showed earlier that for the almost-balanced model the input and output gains are equal; i.e., $\|b_{abi}\|_2 = \|c_{abi}\|_2$. Hence it can be determined from equation (21) that the model is approximately balanced; that is, $W_c \simeq W_o \simeq \Gamma^2$. In terms of the second order grammians, we have $w_c \simeq w_o \simeq \gamma^2$, where $\gamma =$ diag $(\gamma_1, \gamma_2, \ldots, \gamma_p)$ and

$$\gamma_i^2 \simeq \frac{\|b_{abi}\|_2^2}{4\zeta_i\omega_i} \simeq \frac{\|c_{abi}\|_2^2}{4\zeta_i\omega_i}, \tag{25}$$

Example 1. The simple system is shown in Figure 2, with masses $m_1 = m_2 = m_3 = 1$ and stiffness $k_1 = k_2 = k_3 = k_4 = 3$. Its damping matrix is proportional to the stiffness matrix, $D = 0{\cdot}01K$. There is a single input force at mass 3, and three outputs: the displacement and velocity of mass 1, and the velocity of mass 3. The modal model is determined. For this system the natural frequency matrix is $\Omega =$ diag (3.1210, 2.1598, 0.7708), the modal matrix is

$$\Phi = \begin{bmatrix} 0{\cdot}5910 & 0{\cdot}7370 & 0{\cdot}3280 \\ -0{\cdot}7370 & 0{\cdot}3289 & 0{\cdot}5910 \\ 0{\cdot}3280 & -0{\cdot}5910 & 0{\cdot}7370 \end{bmatrix}$$

and the modal damping is $Z = \text{diag}\,(0.0156, 0.0108, 0.0039)$. The modal input and output matrices are $B_m = [0.3280\ \ -0.5910\ \ 0.7370]^T$,

$$C_{mq} = \begin{bmatrix} 0{\cdot}5910 & 0{\cdot}7370 & 0{\cdot}3280 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{mv} = \begin{bmatrix} 0 & 0 & 0 \\ 0{\cdot}5910 & 0{\cdot}7370 & 0{\cdot}3280 \\ 0{\cdot}3280 & -0{\cdot}5910 & 0{\cdot}7370 \end{bmatrix}$$

and therefore, from equation (4),

$$C_m = \begin{bmatrix} 0{\cdot}1894 & 0{\cdot}3412 & 0{\cdot}4255 \\ 0{\cdot}5910 & 0{\cdot}7370 & 0{\cdot}3280 \\ 0{\cdot}3280 & -0{\cdot}5910 & 0{\cdot}7370 \end{bmatrix}.$$

The input and output gains are: $\|b_{m1}\|_2 = 0{\cdot}3280$, $\|c_{m1}\|_2 = 0{\cdot}7020$, $\|b_{m2}\|_2 = 0{\cdot}5910$, $\|c_{m2}\|_2 = 1{\cdot}0044$, $\|b_{m3}\|_2 = 0{\cdot}7370$, $\|c_{m3}\|_2 = 0{\cdot}9120$. The almost-balanced model of the simple structure is determined. The transformation matrix R from equation (9) is $R = \text{diag}\,(0{\cdot}6836, 0{\cdot}7671, 0{\cdot}8989)$. Almost-balanced input and output matrices are obtained from equations (13) and (14):

$$B_{ab} = \begin{bmatrix} 0{\cdot}4798 \\ -0{\cdot}7705 \\ 0{\cdot}8198 \end{bmatrix}, \qquad C_{abq} = \begin{bmatrix} 0{\cdot}4040 & 0{\cdot}5653 & 0{\cdot}2948 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_{abv} = \begin{bmatrix} 0 & 0 & 0 \\ 0{\cdot}4040 & 0{\cdot}5653 & 0{\cdot}2948 \\ 0{\cdot}2242 & -0{\cdot}4534 & 0{\cdot}6625 \end{bmatrix}, \qquad C_{ab} = \begin{bmatrix} 0{\cdot}1294 & 0{\cdot}2617 & 0{\cdot}3825 \\ 0{\cdot}4040 & 0{\cdot}5653 & 0{\cdot}2948 \\ 0{\cdot}2242 & -0{\cdot}4534 & 0{\cdot}6625 \end{bmatrix}.$$

Finally, it is easy to check that the input and output gains are equal; namely, $\|b_{ab1}\|_2 = \|c_{ab1}\|_2 = 0{\cdot}4798$, $\|b_{ab2}\|_2 = \|c_{ab2}\|_2 = 0{\cdot}7705$ and $\|b_{ab3}\|_2 = \|c_{ab3}\|_2 = 0{\cdot}8198$. Also, from equation (25) one obtains $w_{c1} = w_{o1} = 1{\cdot}1821$, $w_{c2} = w_{o2} = 6{\cdot}3628$ and $w_{c3} = w_{o3} = 55{\cdot}8920$, which shows that the model is almost balanced, since the exact Hankel singular values for this system are $\gamma_1^2 = 1{\cdot}1794$, $\gamma_2^2 = 6{\cdot}3736$ and $\gamma_3^2 = 56{\cdot}4212$.

5. STATE SPACE MODELS

Similarly to the second order models, the modal and the almost-balanced state space models are derived.

5.1. MODAL MODELS

Introduce the state vector x_m, which consists of p modal states, $x_m^{\mathrm{T}} = \{x_{m1}^{\mathrm{T}}\ x_{m2}^{\mathrm{T}}\ \dots\ x_{mp}^{\mathrm{T}}\}$. The ith modal state, x_{mi}, is defined as $x_{mi} = \{\omega_i q_{mi}^{\mathrm{T}}\ \dot{q}_{mi}^{\mathrm{T}}\}^{\mathrm{T}}$ (see Gawronski [11]), where q_{mi} and $\dot{q}_{mi}$ are the ith modal displacement and velocity.

5.1.1. *System representation*

The triple (A_m, B_m, C_m) corresponding to the state vector x_m is the modal state space representation of a flexible structure. It has block-diagonal matrix A_m, and the related blocks of B_m and C_m:

$$A_m = \operatorname{diag}(A_{mi}), \qquad B_m = [B_{m1}^{\mathrm{T}} \quad \dots \quad B_{mp}^{\mathrm{T}}]^{\mathrm{T}}, \qquad C_m = [C_{m1} \quad \dots \quad C_{mp}], \tag{26}$$

$i = 1, 2, \dots, p$, where A_{mi}, B_{mi} and C_{mi} are 2×2, $2 \times r$ and $s \times 2$ blocks, respectively:

$$A_{mi} = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\zeta_i\omega_i \end{bmatrix}, \qquad B_{mi} = \begin{bmatrix} 0 \\ b_i \end{bmatrix}, \qquad C_{mi} = \begin{bmatrix} \dfrac{c_{qi}}{\omega_i} & c_{vi} \end{bmatrix}. \tag{27}$$

5.1.2. *Mode representation*

The triple (A_{mi}, B_{mi}, C_{mi}) is the state space representation of the ith mode. It follows from the diagonal form of A_m that the state equation can be written as a set of p equations for each mode:

$$\dot{x}_{mi} = A_{mi}x_{mi} + B_{mi}u, \quad y_i = C_{mi}x_{mi}, \qquad i = 1, \dots, p. \tag{28}$$

The above mode equation is a state space equivalent of the modal equation (6).

Note that $\|B_{mi}\|_2$ and $\|C_{mi}\|_2$ are the input and the output gains of the ith mode, since

$$\|B_{mi}\|_2 = \|b_{mi}\|_2, \qquad \|C_{mi}\|_2 = \|c_{mi}\|_2. \tag{29}$$

Example 2. The state space modal model of form 2 for the system from Example 1 is as follows:

$$A_m = \begin{bmatrix} 0 & 3{\cdot}1210 & 0 & 0 & 0 & 0 \\ -3{\cdot}1210 & -0{\cdot}0974 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2{\cdot}1598 & 0 & 0 \\ 0 & 0 & -2{\cdot}1598 & -0{\cdot}0466 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0{\cdot}7708 \\ 0 & 0 & 0 & 0 & -0{\cdot}7708 & -0{\cdot}0059 \end{bmatrix},$$

$$C_m = \begin{bmatrix} 0{\cdot}1894 & 0 & 0{\cdot}3412 & 0 & 0{\cdot}4255 & 0 \\ 0 & 0{\cdot}5910 & 0 & 0{\cdot}7370 & 0 & 0{\cdot}3280 \\ 0 & 0{\cdot}3280 & 0 & -0{\cdot}5910 & 0 & 0{\cdot}7370 \end{bmatrix}$$

and $B_m^{\mathrm{T}} = [0\ \ 0{\cdot}3280\ \ 0\ \ -0{\cdot}5910\ \ 0\ \ 0{\cdot}7370]$. The input and output gains are $\|B_{m1}\|_2 = 0{\cdot}3280$, $\|C_{m1}\|_2 = 0{\cdot}7020$, $\|B_{m2}\|_2 = 0{\cdot}5910$, $\|C_{m2}\|_2 = 1{\cdot}0044$, $\|B_{m3}\|_2 = 0{\cdot}7370$ and $\|C_{m3}\|_2 = 0{\cdot}9120$.

5.2. ALMOST-BALANCED MODEL

The almost-balanced state, x_{ab}, is obtained from the modal state x_m using the transformation $x_m = R_{ab}x_{ab}$, where R_{ab} is a diagonal matrix,

$$R_{ab} = \operatorname{diag}(R_{abi}), \quad R_{abi} = r_iI_2, \qquad i = 1, \dots, p \tag{30}$$

and r_i is given by equation (9).

5.2.1. *System representation*

The triple (A_{ab}, B_{ab}, C_{ab}) is the almost-balanced state space representation. The matrix A_{ab} is the same as the modal matrix, since $A_{ab} = R_{ab}^{-1}A_mR_{ab} = R_{ab}^{-1}R_{ab}A_m = A_m$. The matrices

B_{ab} and C_{ab} are obtained by scaling the modal matrices B_m and C_m; i.e., $B_{ab} = R_{ab}^{-1} B_m$ and $C_{ab} = C_m R_{ab}$.

5.2.2. *Mode representation*

Similarly to the modal representations, the almost-balanced state vector x_{ab} is divided into p modal states, $x_{ab}^{\mathrm{T}} = \{x_{ab1}^{\mathrm{T}} \; x_{ab2}^{\mathrm{T}} \; \ldots \; x_{abp}^{\mathrm{T}}\}$. With each state,

$$x_{abi} = x_{mi}/r_i, \tag{31a}$$

the mode state space representation $(A_{abi}, B_{abi}, C_{abi})$ and the state equation

$$\dot{x}_{abi} = A_{abi} x_{abi} + B_{abi} u, \qquad y_i = C_{abi} x_{abi} \tag{31b}$$

is associated.

In the almost-balanced state space representation $(A_{abi}, B_{abi}, C_{abi})$ the input and output gains are equal:

$$\|B_{abi}\|_2 = \|C_{abi}\|_2 = \sqrt{\|B_{mi}\|_2 \|C_{mi}\|_2}, \tag{32}$$

which can be proven by applying the transformation R_{ab} as in equation (30) to the input and output matrices.

The above properties show that the system matrix A_{ab} of the almost-balanced representation is independent of the actuator and sensor location, and that the orientation of the almost-balanced and modal co-ordinates is identical, although of different scale.

5.2.3. *Controllability and observability grammians*

The grammians of the almost-balanced representation are in the form

$$W_c \simeq \operatorname{diag}(W_{ci}), \qquad W_{ci} = w_{ci} I_2, \qquad W_o \simeq \operatorname{diag}(W_{oi}), \qquad W_{oi} = w_{oi} I_2, \tag{33}$$

where w_{ci} and w_{oi} are given by equation (21). Since $w_{ci} \simeq w_{oi}$, the system is almost balanced; that is, $\Gamma^2 \simeq W_c \simeq W_o$, where $\Gamma \simeq \operatorname{diag}(\gamma_1, \gamma_1, \gamma_2, \gamma_2, \ldots, \gamma_{n_d}, \gamma_{n_d})$, and

$$\gamma_i^2 \simeq \frac{\|B_{abi}\|_2^2}{4\zeta_i \omega_i} = \frac{\|C_{abi}\|_2^2}{4\zeta_i \omega_i}. \tag{34}$$

Example 3. The almost-balanced representation of the simple system from Example 1 is obtained. Using the modal state space representation, as in Example 2, one finds the transformation matrix R_{ab} as in equation (30); $R_{ab} = \operatorname{diag}(0{\cdot}6836,\ 0{\cdot}6836,\ 0{\cdot}7671,\ 0{\cdot}7671,\ 0{\cdot}8989,\ 0{\cdot}8989)$. The state matrix is $A_{ab} = A_m$, while the B_{ab} and C_{ab} are $B_{ab}^{\mathrm{T}} = [0,\ 0{\cdot}4798,\ 0,\ -0{\cdot}7707,\ 0,\ 0{\cdot}8198]$ and

$$C_{ab} = \begin{bmatrix} 0{\cdot}1294 & 0 & 0{\cdot}2617 & 0 & 0{\cdot}3825 & 0 \\ 0 & 0{\cdot}4040 & 0 & 0{\cdot}5653 & 0 & 0{\cdot}2948 \\ 0 & 0{\cdot}2242 & 0 & -0{\cdot}4534 & 0 & 0{\cdot}6625 \end{bmatrix}.$$

In this representation equation (32) holds; namely, $\|B_{ab1}\|_2 = \|C_{ab1}\|_2 = 0{\cdot}4798$, $\|B_{ab2}\|_2 = \|C_{ab2}\|_2 = 0{\cdot}7705$ and $\|B_{ab3}\|_2 = \|C_{ab3}\|_2 = 0{\cdot}8198$. The grammians obtained for this model are almost equal; i.e., $\Gamma^2 \simeq W_o \simeq W_c = \operatorname{diag}(1{\cdot}1817,\ 1{\cdot}1817,\ 6{\cdot}3627,\ 6{\cdot}3627,\ 56{\cdot}5585,\ 56{\cdot}5585)$.

6. MODE AND SYSTEM NORMS

For flexible systems in the almost-balanced representation, the Hankel, H_2 and H_∞ norms are determined in terms of their parameters. The norms serve as system measures and are used later in model reduction and in the actuator/sensor placement procedures.

6.1. HANKEL NORM

The Hankel norm of a system is a measure of the effect of its past input on its future output, or the amount of energy stored in, and subsequently retrieved from the system [25, p. 103], and given by $\|G\|_h^2 = \lambda_{max}(W_o W_c)$, where $\lambda_{max}(\cdot)$ denotes the largest eigenvalue.

6.1.1. *Mode norm*

Consider the ith almost-balanced mode in the state space form $(A_{abi}, B_{abi}, C_{abi})$, or the corresponding second order form $(\omega_i, \zeta_i, b_{abi}, c_{abi})$. Its Hankel norm is

$$\|G_{abi}\|_h = \gamma_i^2. \tag{35}$$

6.1.2. *System norm*

The Hankel norm of the system is the largest norm of its modes; i.e.,

$$\|G_{ab}\|_h \simeq \max_i \|G_{abi}\|_h = \gamma_{max}^2, \tag{36}$$

where γ_{max} is the largest Hankel singular value of the system.

6.2. H_2 NORM

Let (A, B, C) be a system state space representation, and let $G(\omega) = C(\mathrm{j}\omega I - A)^{-1}B$ be its transfer function. The H_2 norm of the system is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(\omega)|^2\,\mathrm{d}\omega, \qquad \text{or as} \qquad \|G\|_2^2 = \mathrm{tr}\,(C^{\mathrm{T}}CW_c), \tag{37}$$

where W_c is a solution of the Lyapunov equation (1).

6.2.1. *Mode norm*

Let $G_{abi}(\omega) = C_{abi}(\mathrm{j}\omega I - A_{abi})^{-1}B_{abi}$ be the transfer function of the ith almost-balanced mode. From the definition of the norm, one obtains

$$\|G_{abi}\|_2 = \sqrt{\mathrm{tr}\,(C_{abi}^{\mathrm{T}}C_{abi}W_{cabi})} = \frac{\|B_{abi}\|_2\|C_{abi}\|_2}{\sqrt{2\Delta\omega_i}} \simeq \frac{\|B_{abi}\|_2^2}{\sqrt{2\Delta\omega_i}} \simeq \frac{\|C_{abi}\|_2^2}{\sqrt{2\Delta\omega_i}} \simeq \sqrt{\Delta\omega_i}\gamma_i^2, \tag{38}$$

where $\Delta\omega_i = 2\zeta_i\omega_i$ is the half-power frequency [26, pp. 157 and 165]. For the norm of the second order almost-balanced representation $(\omega_i, \zeta_i, b_{abi}, c_{abi})$, replace B_{abi} and C_{abi} with b_{abi} and c_{abi}, respectively. Note also that $\|G_{abi}\|_2$ is the modal cost of Skelton [6].

6.2.2. *System norm*

Let $G_{ab}(\omega) = C_{ab}(\mathrm{j}\omega I - A_{ab})^{-1}B_{ab}$ be the transfer function of the almost-balanced structure. Since its controllability grammian W_{cab} is diagonally dominant, its H_2 norm is

$$\|G_{ab}\|_2^2 = \mathrm{tr}\,(C_{ab}^{\mathrm{T}}C_{ab}W_{cab}) \cong \sum_{i=1}^{p} \mathrm{tr}\,(C_{abi}^{\mathrm{T}}C_{abi}W_{cabi}) = \sum_{i=1}^{p} \|G_{abi}\|_2^2; \tag{39}$$

i.e. the system H_2 norm is, approximately, the root-mean-square (r.m.s.) sum of the modal norms:

$$\|G_{ab}\|_2 \simeq \sqrt{\sum_{i=1}^{p} \|G_{abi}\|_2^2}. \tag{40}$$

Example 4: H_2 norms of modes and the system. For the simple system in the almost-balanced co-ordinates (second order, Example 1; or state, Example 3) one finds the

H_2 mode norms, from equation (38): $\|G_{ab1}\|_2 = 0{\cdot}5219$, $\|G_{ab2}\|_2 = 1{\cdot}9420$ and $\|G_{ab3}\|_2 = 6{\cdot}1289$. According to equation (40), the system norm is $\|G_{ab}\|_2 = 6{\cdot}4504$.

6.3. H_∞ NORM

The H_∞ norm of the system, (A, B, C), is defined as

$$\|G\|_\infty = \sup_\omega \sigma_{max}(G(\omega)), \tag{41}$$

where G is the system transfer function and $\sigma_{max}(G(\omega))$ is the largest singular value of G.

6.3.1. *Mode norm*

Consider the ith almost-balanced mode $(A_{abi}, B_{abi}, C_{abi})$, or $(\omega_i, \zeta_i, b_{abi}, c_{abi})$. Its H_∞ norm is estimated as

$$\|G_{abi}\|_\infty \simeq 2\gamma_i^2. \tag{42}$$

In order to prove it, note that the largest amplitude of the mode is approximately at the ith natural frequency; thus,

$$\|G_{abi}\|_\infty \simeq \sigma_{max}(G_{abi}(\omega_i)) = \frac{\sigma_{max}(C_{abi}B_{abi})}{2\zeta_i\omega_i} = \frac{\|B_{abi}\|_2\|C_{abi}\|_2}{2\zeta_i\omega_i} \simeq 2\gamma_i^2. \tag{43}$$

6.3.2. *System norm*

Due to the near-independence of the modes, the system H_∞ norm is the largest of the mode norms; i.e.,

$$\|G_{ab}\|_\infty \simeq \max_i \|G_{abi}\|_\infty \simeq 2\gamma_{max}^2, \tag{44}$$

where γ_{max} is the largest Hankel singular value of a structure.

Example 5: H_∞ norms of modes and systems. For the simple system in almost-balanced co-ordinates (second order, Example 1; or state, Example 3) one finds the H_∞ mode norms, from equation (42), $\|G_{ab1}\|_\infty \simeq 2\gamma_1^2 = 2{\cdot}3635$, $\|G_{ab2}\|_\infty \simeq 2\gamma_2^2 = 12{\cdot}7253$, $\|G_{ab3}\|_\infty \simeq 2\gamma_3^2 = 113{\cdot}1169$, and the system H_∞ norm is the largest mode norm, $\|G_{ab}\|_\infty \simeq \max_i \|G_{abi}\|_\infty = \|G_{ab3}\|_\infty = 113{\cdot}1169$ $i = 1, 2, 3$. The actual H_∞ norm, determined from equation (44), is $\|G_{ab}\|_\infty = 113{\cdot}1170$.

Example 6. Consider a truss, presented in Figure 3. For this structure, $l_1 = 20$ cm, $l_2 = 30$ cm, and each truss has a cross-sectional area of 1 cm^2, a Young's modulus of $2{\cdot}1 \times 10^7$ N/cm^2, and a mass density of $7{\cdot}85 \times 10^{-3}$ kg/cm^3. Vertical control forces are applied at nodes $n7$ and $n8$, and the output rates are measured in the vertical direction at nodes $n3$ and $n4$. The system has 26 states (13 modes), two inputs and two outputs. For this structure the natural frequencies, H_2 norms and H_∞ norms of each mode are given in Table 1. The exact Hankel singular values, and the approximate ones, obtained from equations (34) or (25), are shown in Figure 4. A good coincidence between the exact and the approximate values is observed.

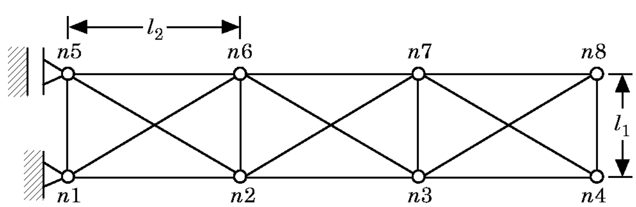

Figure 3. A truss.

TABLE 1

Mode norms of the truss

Natural frequency (rad/s)	$\|G_i\|_2$	$\|G_i\|_\infty$
232·1	0·25581	0·11683
927·1	0·18616	0·09296
1304·8	0·00006	0·00003
1785·2	0·32224	0·16953
3359·2	0·09598	0·04627
3457·4	0·06527	0·03235
3648·9	0·14568	0·06646
3890·2	0·00099	0·00048
4089·9	0·04971	0·02452
4220·4	0·41664	0·21239
4237·7	0·04989	0·02543
4852·2	0·02623	0·01251
5479·1	0·01100	0·00487

7. MODEL REDUCTION

Model reduction is a part of a dynamic analysis of structures. Typically, a model with a large number of degrees of freedom, such as one developed for static analysis, causes numerical difficulties when applied to dynamic analysis, not to mention computational cost. On the other hand, in system identification the order of the identified system is determined by the reduction of the oversized model. Finally, in structural control the complexity and performance of a controller depends on the rational order reduction of the structural model. In all cases, the reduction is a crucial part of analysis and design.

A reduced order model of a linear system is obtained here by truncating appropriate modes of the almost-balanced model. Denote x_{ab} the almost-balanced state vector of p modes ($n = 2p$ states), and (A_{ab}, B_{ab}, C_{ab}) is the almost-balanced representation. Let x_{ab} be partitioned, $x_{ab}^T = [x_{abr}^T \;\; x_{abt}^T]$, where x_{abr} is the vector of the retained states, and x_{abt} is a vector of the truncated states. If there are $k < p$ retained modes, x_{abr} is a vector of $2k$ states, and x_{abt} is a vector of $2(p-k)$ states. The reduced model is obtained by deleting the last $2(p-k)$ rows of A_{ab}, B_{ab} and the last $2(p-k)$ columns of A_{ab}, C_{ab}. The problem is how to order states so that the retained states x_{abr} will be the best reproduction of the full system response. The choice depends on the definition of the reduction index.

The error between the full and reduced system is used for the reduction evaluation. We have taken two approaches in the evaluation of the reduction error. The first approach is based on the H_2 norm, and is connected to the Skelton reduction method [6]. The second is based on the H_∞ and Hankel norm, and is connected to the Moore reduction method [4].

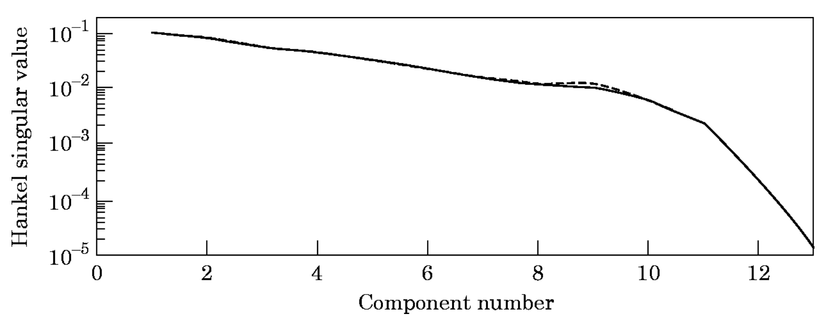

Figure 4. Exact (solid line) and approximate (dashed line) Hankel singular values of the truss.

7.1. H_2 MODEL REDUCTION

The H_2 error is defined as

$$e_2 = \|G - G_r\|_2, \tag{45}$$

where G is the transfer function of the full model and G_r is the transfer function of the reduced model. Its interpretation is as follows. Let the impulse input be applied to the full and reduced system; y is the impulse response of the full system and y_r is the impulse response of the reduced system. Then $e_2 = \|y - y_r\|_2$; thus e_2 is the root-mean-square (r.m.s.) measure of the output error due to impulse input.

The squares of the mode norm are additive—see equation (40)—therefore the norm of the reduced system with k modes is the r.m.s. sum of the mode norms,

$$\|G_r\|_2^2 \simeq \sum_{i=1}^{k} \|G_i\|_2^2. \tag{46}$$

Thus, the reduction error is

$$e_2^2 \simeq \|G\|_2^2 - \|G_r\|_2^2 \simeq \sum_{i=k+1}^{p} \|G_i\|_2^2. \tag{47}$$

It is clear from the above equation that the near-optimal reduction is obtained if the truncated mode norms $\|G_i\|_2$ for $i = k+1, \ldots, p$ are the smallest ones. Therefore, the almost-balanced mode vector is rearranged, starting from the mode with the largest H_2 norm and ending with the mode with the smallest norm. Truncation of the last $p - k$ modes will give, in this case, a near-optimal reduced model of order k.

7.2. HANKEL AND H_∞ MODEL REDUCTION

It can be seen from equations (35) and (42) that the H_∞ norm is approximately twice the Hankel norm; hence the reduction using one of those norms is identical with the reduction using the other one. Therefore, here we consider the H_∞ reduction only.

The H_∞ reduction error is defined as

$$e_\infty = \|G - G_r\|_\infty. \tag{48}$$

It was shown by Glover [27] that the upper limit of the H_∞ reduction error is

$$e_\infty = \|G - G_r\|_\infty \leqslant \sum_{i=k+1}^{n} \|G_i\|_\infty. \tag{49}$$

However, for the flexible structures in the almost-balanced co-ordinates the error can be estimated less conservatively. In this case the transfer function is approximately a sum of its modes; that is,

$$G \cong \sum_{i=1}^{p} G_i, \qquad G_r \simeq \sum_{i=1}^{k} G_i; \tag{50}$$

thus

$$G - G_r \simeq \textstyle\sum_{i=k+1}^{p} G_i = G_t, \tag{51}$$

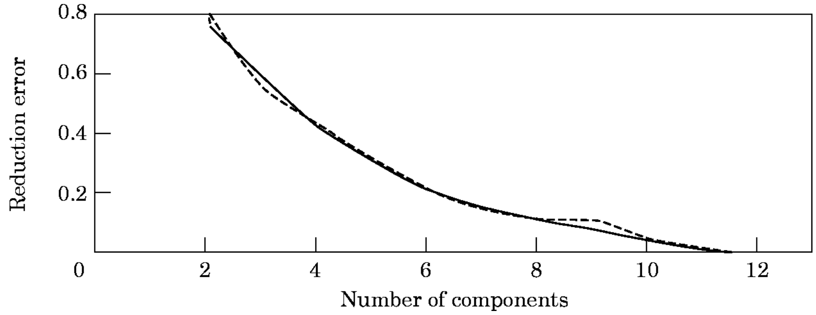

Figure 5. H_2 (solid line) and H_∞ (dashed line) normalized errors of the truss.

where G_t is the transfer function of the truncated part. Therefore,

$$e_\infty = \|G - G_r\|_\infty \simeq \|G_t\|_\infty \simeq \|G_{k+1}\|_\infty. \tag{52}$$

It is clear that the near-optimal reduction is obtained if the H_∞ norms of the truncated modes are the smallest ones. Moore [4] and Glover [27] showed that the reduced model is stable.

Example 7. A simple system is considered, as in Example 1. For this system the H_∞ modal norms are obtained from equations (34) and (42): $\|G_1\|_\infty \simeq 6{\cdot}7586$ (mode of the natural frequency 1·3256 rad/s), $\|G_2\|_\infty \simeq 4{\cdot}9556$ (mode of the natural frequency 2·4493 rad/s), $\|G_3\|_\infty \simeq 2{\cdot}6526$ (mode of the natural frequency 3·200 rad/s). The H_2 mode norms are as follows: $\|G_1\|_2 \simeq 3{\cdot}2299$, $\|G_2\|_2 \simeq 3{\cdot}3951$, $\|G_3\|_2 \simeq 0{\cdot}5937$. The reduction errors after reduction of the last mode (of frequency 3·200 rad/s) are $e_2 = 0{\cdot}7959$ and $e_\infty = 3{\cdot}5182$.

Example 8. Consider a truss, as in Example 6. Its model has been reduced in the almost-balanced co-ordinates using the H_2 and H_∞ norms. The normalized errors $\delta_2 = \varepsilon_2/\|G\|_2$ and $\delta_\infty = \varepsilon_\infty/\|G\|_\infty$ were computed using equations (45) and (48), respectively. The plot of the errors with respect to the number of modes of the reduced model is given in Figure 5. Both errors are close to each other, and the plot indicates that an error smaller than 0·1 is obtained for the reduced models which contain eight or more modes. The accuracy of the estimation of the H_2 reduction error is shown in Figure 6, where the solid line refers to the accurate reduction errors obtained from the H_2 norm definition, and the dashed line refers to the estimated reduction errors obtained from equation (47). The plot shows that the accuracy of error estimation is satisfactory.

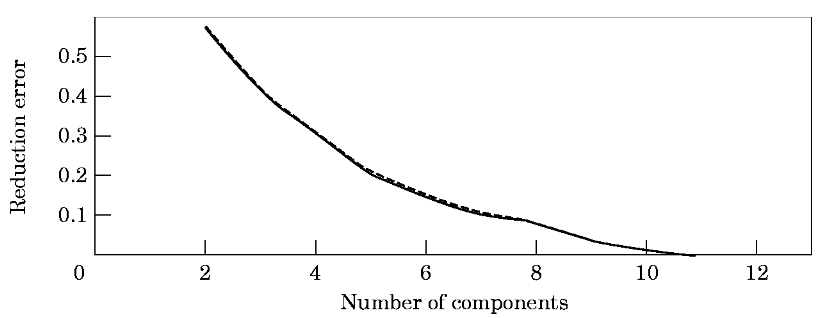

Figure 6. Actual (solid line) and estimated H_2 (dashed line) normalized errors for the truss.

8. ACTUATOR AND SENSOR PLACEMENT

For the purposes of structural testing and control, it is useful to investigate possible sensor and/or actuator locations, and to evaluate their impact on the dynamic test results, or on the closed loop performance. Given a large set of sensors and actuators, the placement problem consists of determining the locations of a smaller subset such that the H_2 or H_∞ norm of the subset is close to the original set. Here the placement problem is solved in the almost-balanced co-ordinates using the previously derived properties. In this case a comparatively simple methodology of choice of a small subset of sensors and/or actuators from a large set of possible locations is proposed.

Let **R** and **S** be the sets of the candidate actuator and sensor locations, respectively. These are chosen in advance to be all allowable locations of actuators, of population R, and all allowable locations of sensors, of population S. The placement of r actuators within the given **R** actuator candidate locations, and the placement s sensors within the given **S** sensor candidate locations, are considered. Of course, the number of candidate locations is larger than the number of actuators or sensors; i.e., $r < R$ and $s < S$.

8.1. ADDITIVE PROPERTY OF THE MODAL NORMS

8.1.1. H_2 *norm*

Now consider a flexible structure in the almost-balanced representation. The H_2 norm of the ith mode is given by equation (38), where $\|B_{abi}\|_2$ and $\|C_{abi}\|_2$ are the input and output gains of the ith mode. The input and output matrices are $B_{abi} = [B_{abi1}\ \ B_{abi2}\ \ \ldots\ \ B_{abiR}]$, $C_{abi} = [C^{\mathrm{T}}_{ab1i}\ \ C^{\mathrm{T}}_{ab2i}\ \ \ldots\ \ C^{\mathrm{T}}_{abiS}]^{\mathrm{T}}$, and B_{abij} is the 2×1 block of the jth input, while C_{abji} is the 1×2 block of the jth output. In this notation one obtains

$$\|B_{abi}\|_2^2 = \sum_{j=1}^{R} \|B_{abij}\|_2^2, \qquad \|C_{abi}\|_2^2 = \sum_{k=1}^{S} \|C_{abki}\|_2^2. \tag{53}$$

Introducing equation (53) to equation (38), one obtains

$$\|G_i\|_2^2 \simeq \sum_{j=1}^{R} \|G_{ij}\|_2^2, \qquad \text{or} \qquad \|G_i\|_2^2 \simeq \sum_{k=1}^{S} \|G_{ik}\|_2^2, \tag{54a}$$

where

$$\|G_{ij}\|_2 = \frac{\|B_{abij}\|_2 \|C_{abi}\|_2}{2\sqrt{\zeta_i \omega_i}}, \qquad \|G_{ik}\|_2 = \frac{\|B_{abi}\|_2 \|C_{abki}\|_2}{2\sqrt{\zeta_i \omega_i}} \tag{54b}$$

are the H_2 norms of the ith mode with the jth actuator only, or the ith mode with the kth sensor only. Equation (54a) shows that H_2 norm of a mode with a set of actuators (sensors) is the r.m.s. sum of the H_2 norms of this mode wth a single actuator (sensor).

8.1.2. H_∞ *norm*

A similar relationship can be obtained for the H_∞ norm. From equation (42), one obtains

$$\|G_i\|_\infty \simeq \frac{\|B_{abi}\|_2 \|C_{abi}\|_2}{2\zeta_i \omega_i}. \tag{55}$$

Introducing equation (53) to equation (55), one obtains

$$\|G_i\|_\infty^2 \simeq \sum_{j=1}^{R} \|G_{ij}\|_\infty^2, \qquad \text{or} \qquad \|G_i\|_\infty^2 \simeq \sum_{k=1}^{S} \|G_{ik}\|_\infty^2, \tag{56a}$$

where

$$\|G_{ij}\|_{\infty} = \frac{\|B_{abij}\|_2 \|C_{abi}\|_2}{2\zeta_i \omega_i}, \qquad \|G_{ik}\|_{\infty} = \frac{\|B_{abi}\|_2 \|C_{abki}\|_2}{2\zeta_i \omega_i} \tag{56b}$$

are the H_∞ norms of the ith mode with the jth actuator only, or the ith mode with the kth sensor only. Equation (56a) shows that the H_∞ norm of a mode with a set of actuators (sensors) is the r.m.s. sum of the H_∞ norms of this mode with a single actuator (sensor).

8.2. PLACEMENT INDICES

Two similar problems can be distinguished: actuator placement and sensor placement. Due to their similarity, the actuator placement problem only is considered.

8.2.1. H_2 *placement indices*

Denote by G the transfer function of the system with all R actuators. The placement index σ_{ki} that evaluates the kth actuator at the ith mode in terms of the two-norm is defined with respect to the all modes and all admissible actuators:

$$\sigma_{ki} = \|G_{ki}\|_2 / \|G\|_2, \qquad k = 1, \ldots, R, \quad i = 1, \ldots, p. \tag{57}$$

In applications, it is convenient to represent the two-norm placement indices as a placement matrix:

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} & \cdots & \sigma_{1R} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} & \cdots & \sigma_{2R} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{i1} & \sigma_{i2} & \cdots & \sigma_{ik} & \cdots & \sigma_{iR} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pk} & \cdots & \sigma_{pR} \end{bmatrix} \Leftarrow i\text{th mode.} \tag{58}$$

$\Uparrow$
kth actuator

8.2.2. H_∞ *placement indices*

Similarly to the two-norm index, the placement index σ_{ki} that evaluates the kth actuator at the ith mode in terms of the infinity-norm is defined in relation to all modes and all admissible actuators:

$$\sigma_{ki} = \|G_{ki}\|_\infty / \|G\|_\infty, \qquad k = 1, \ldots, R, \quad i = 1, \ldots, p. \tag{59}$$

Using the above indices, one introduces the infinity-norm placement matrix Σ of similar structure as in equation (58).

8.2.3. *Actuator and mode indices*

The placement matrix Σ gives an insight into the placement properties of each actuator, since the placement index of the kth actuator is determined as the root-mean-square (r.m.s.) sum of the kth column of Σ. The vector of the actuator placement indices is defined as $\sigma_a = [\sigma_{a1} \;\; \sigma_{a2} \;\; \ldots \;\; \sigma_{aR}]^{\mathrm{T}}$, and its kth entry,

$$\sigma_{ak} = \sqrt{\sum_{i=1}^{p} \sigma_{ik}^2}, \qquad k = 1, \ldots, R, \tag{60}$$

TABLE 2

Placement indices, σ_{ai}

	Actuator							
	1	2	3	4	5	6	7	8
Example 9	0·3671	0·4944	0·3328	0·4460	0·2626	0·4921	—	—
Example 10	0·0967	0·3516	0·1260	0·5932	0·0936	0·3519	0·1219	0·5932

is the placement index of the kth actuator. It is the r.m.s. sum of the kth actuator indexes over all modes. The actuator placement index, σ_{ak}, is a non-negative contribution of the kth actuator at all modes to the controllability and observability properties of the structure.

Similarly, the vector of the mode indices can be defined as $\sigma_m = [\sigma_{m1} \ \ \sigma_{m2} \ \ldots \ \sigma_{mn_d}]^{\mathrm{T}}$, and its ith entry is

$$\sigma_{mi} = \sqrt{\sum_{k=1}^{p} \sigma_{ik}^2}, \qquad i = 1, \ldots, n_d \tag{61}$$

is the index of the ith mode. It is the r.m.s. sum of the ith mode indexes over all actuators. The modal index, σ_{mi}, is a non-negative contribution of the ith mode for all actuators to the controllability and observability properties of the structure.

From the above properties it follows that the index σ_{ak} characterizes the importance of the kth actuator; thus it serves as the actuator placement index. Namely, the actuators with small index σ_{ak} can be removed as the least significant ones. Note also that the mode index σ_{mi} can also be used either as a placement index in cases in which modes of the required controllability and observability level are sought, or as a reduction index. Indeed, it characterizes the significance of the ith balanced mode for the given locations of sensors and actuators. The controllability and observability of the least significant modes (those with the small index σ_{mi}) should either be enhanced by the reconfiguration of the actuators or sensors, or be eliminated (i.e., the system order is reduced).

Example 9. The truss from Figure 3 is considered. Its outputs include vertical displacements at nodes $n7$ and $n8$, respectively. The following actuators are considered: (1) force in the bar connecting nodes $n2$ and $n1$; (2) force in the bar connecting nodes $n3$ and $n2$; (3) force in the bar connecting nodes $n6$ and $n5$; (4) force in the bar connecting nodes $n7$ and $n6$; (5) force in the bar connecting nodes $n2$ and $n6$; and (6) force in the bar connecting nodes $n3$ and $n7$. The task is to find the two best inputs within the given six candidates using the infinity-norm indices.

The placement indices σ_{ai}, $i = 1, \ldots, 6$ of each actuator are given in Table 2. The indices from the table point out that the second actuator (connecting nodes $n3$ and $n2$) and the sixth actuator (connecting nodes $n7$ and $n3$) are the most appropriately located.

Example 10. The same truss is considered, with the same outputs. The eight candidate actuator locations are given: (1) horizontal force at node $n3$; (2) vertical force at node $n3$; (3) horizontal force at node $n4$; (4) vertical force at node $n4$; (5) horizontal force at node $n7$; (6) vertical force at node $n7$; (7) horizontal force at node $n8$; and (8) vertical force at node $n8$. The task is to find the best two inputs within the candidate locations.

The actuator indices σ_{ai} obtained from equation (60) are presented in Table 2. The results indicate that the locations 4 (vertical force at node $n4$), and 8 (vertical force at node $n8$) are the best choices.

TABLE 3

Second order and state space models of an almost-balanced mode

	Second order model	State space model
Equation	Equation (16b)	Equation (31b)
Co-ordinates	q_{abi}, equation (16a)	x_{abi}, equation (31a)
Transformation from modal co-ordinates	$1/r_i$, equation (9)	I_2/r_i, equation (30)
Gains	Equation (18)	Equation (32)
Grammians	Equation (20)	Equation (33)
Hankel singular values	Equation (25)	Equation (34)
H_2 norm	Equation (38)	Equation (38)
H_∞ norm	Equation (42)	Equation (42)

9. CONCLUSIONS

Structural engineers use predominantly second order models in structural analysis. Some quantities, such as controllability and observability grammians, system norms, and system properties such as balanced systems, have been developed by control engineers for the purposes of control system design. We have shown that these quantities and properties can be interpreted for the second order system by introducing proper scaling of the modal co-ordinates. Moreover, we have shown that the modal H_2, H_∞ and Hankel norms are obtained from the system modal parameters and its input and output gains. Also, it was demonstrated that the system H_2 and H_∞ norms are r.m.s. sums of the mode norms. The relationship between the second order almost-balanced modes and state space almost-balanced modes is summarized in Table 3.

The almost-balanced representation was used to model reduction of flexible structures, as well as to place the actuators (or sensors). It was shown that the almost-balanced co-ordinates allow for near-optimal model reduction of structures, in the sense of H_2, H_∞ and Hankel norms. It was also shown that one can place the system actuators and sensors to satisfy specified H_2 and H_∞ norm criteria. This is done by using the superposition of a modal norms of a single actuator (sensor) to obtain the modal norm of a set of actuators or sensors.

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